Scissors Congruence for Certain k-polygons

By

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Abstract

It has been proved that any two polygons having the same area are scissors congruent by Bolyai in 1832 and by Gerwien in 1833, respectively. It is well known that the concepts of congruence and scissors congruence are different for the set of polygons in the Euclidean plane. Let C be a unit circle divided into n parts equally. We denote the set of ends of these parts on C by $S = \{P_0, P_1, \ldots, P_{n-1}\}$. Let $\wp_k(n)$ be the set of all k-polygons inscribed in C, where the vertices are taken from S. In this paper, we shall investigate the relations of the concepts of congruence and scissors congruence in this special set of k-polygons $\wp_k(n)$.

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Introduction

Let us recall that two figures F and G in the Euclidean plane are said to be *congruent* if F can be moved without changing its size or shape so as to coincide with the other figure G. If two figures F and G are congruent, we shall express $F \cong G$ in symbols. It follows that congruence is an equivalence relation on the set of all the figures in the Euclidean plane.
On the other hand, two figures F and G in the Euclidean plane are said to be *scissors congruent* (or *equidecomposable* or also *congruent by dissection*) if the

figure F can be suitably decomposed into a finite number of pieces which can be reassembled to give the figure G. We shall express $F \sim G$, if two figures F and G are scissors congruent. It is easy to verify that the scissors congruence is also an equivalence relation on the set of all the figures in the Euclidean plane. Let us denote the area of a figure F by $A(F)$. It is easy to see that, if $F \cong G$ then $F \sim G$ and there exist figures F and G which are $F \ncong G$ but $F \sim G$. It also follows that if $F \sim G$ then $A(F) = A(G)$. We shall restrict ourselves to the set of all the polygons in the Euclidean plane for a while. Now we shall introduce the following notation. F denotes a polygon and $F_1, F_2, \dots, F_r$ denote polygons in F, which satisfy

(1) the interiors of F_i and F_j are disjoint for any $i \neq j$ $(1 \leq i < j \leq r)$, and

(2) $F = F_1 \cup F_2 \cup \ldots \cup F_r$.

Then we shall write

$$F = F_1 + F_2 + \cdots + F_r.$$

Now we can write down the conditions of scissors congruence as follows. Two figures $F \sim G$, if and only if $F = F_1 + \cdots + F_r$ and $G = G_1 + \cdots G_r$, where $\{F_i\}$ and $\{G_i\}$ are suitable decompositions of F and G with $F_i \cong G_i$ for $(1 \leq i \leq r)$. It was found independently by F. Bolyai (in 1832) and by P. Gerwein (in 1833) that two polygons have the same area if and only if they are scissors congruent (see [2] or [6]).

We shall recall the concept of *equicomplementability* as follows. Two figures F and G are *equicomplementable* if there exist suitable figures $F_1, \dots, F_s$ and $G_1, \dots, G_s$ with $F_i \cong G_i$ $(1 \leq i \leq s)$ such that

$$F + (F_1 + \cdots + F_s) = G + (G_1 + \cdots + G_s).$$

Bolyai and Gerwein theorem asserts that *equality of area*, *scissors congruence* and *equicomplementability* are equivalent properties of polygons, that is:

Proposition 1 (Bolyai-Gerwien). *The following conditions of two polygons F and G are equivalent*

(1) *F and G have the same area,*
(2) *F and G are scissors congruent,*
(3) *F and G are equicomplementable.*

Let C be a unit circle divided into n parts equally. We denote the ends of these parts on C by $P_0, P_1, \dots, P_{n-1}$ and denote the set of ends $\{P_0, P_1, \dots, P_{n-1}\}$ by S. For any natural number k, we denote by $\wp_k(n)$ the set of all $k-$polygons, where the vertices $P_{i_1}, P_{i_2}, \dots, P_{i_k}$ $(0 \leq i_1 < i_2 < \cdots < i_k \leq n-1)$ are

taken from S. Put $a_1 = i_2 - i_1, a_2 = i_3 - i_2, \ldots, a_{k-1} = i_k - i_{k-1}$ and $a_k = n + i_1 - i_k$. Let O be the center of the given circle C. Then the central angles of the $k-$polygon $P_{i_1}P_{i_2}\cdots P_{i_k}$ are $\angle P_{i_j}OP_{i_{j+1}} = \dfrac{2a_j\pi}{n}$ $(1 \leq j \leq k-1)$ and $\angle P_{i_k}OP_{i_1} = \dfrac{2a_k\pi}{n}$.
Then $a_1, a_2, \ldots, a_k$ satisfies $a_1, a_2, \ldots, a_k \geq 1$ and $a_1 + a_2 + \cdots + a_k = n$. In the following, we shall denote the $k-$polygon $P_0P_{a_1}P_{a_1+a_2}\cdots P_{a_1+a_2+\cdots+a_{k-1}}$ by $P(a_1, a_2, \ldots, a_k)$. Then any $k-$polygon $P_{i_1}P_{i_2}\cdots P_{i_k}$ is congruent to the $k-$polygon $P(a_1, a_2, \ldots, a_k)$, where $a_1 = i_2 - i_1, \ldots, a_{k-1} = i_k - i_{k-1}$ and $a_k = n + i_1 - i_k$ as above.

Results for $\wp_k(n)$

Now we shall consider the relations of two concepts congruence ($\cong$) and scissors congruence ($\sim$) in the special set of polygons $\wp_k(n)$. For any $k-$polygon $P(a_1, a_2, \ldots, a_k)$, we shall denote the triangle $OP_{i_j}P_{i_{j+1}}$ with $\angle P_{i_j}OP_{i_{j+1}} = \dfrac{2a_j\pi}{n}$ by $\triangle(a_j)$. We note that, if $a_j > n/2$, we may consider the area $A(\triangle(a_j)) = \sin(2a_j\pi/n)/2 < 0$. We can express

$$P(a_1, a_2, \ldots, a_k) = \triangle(a_1) + \triangle(a_2) + \cdots + \triangle(a_k).$$

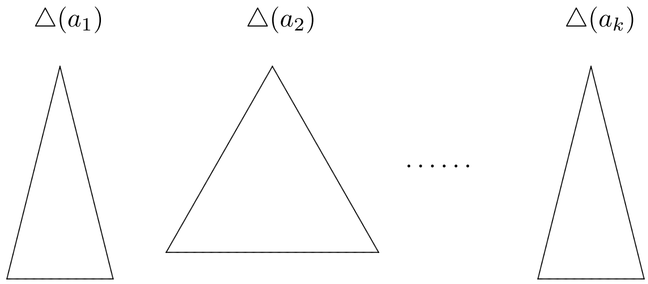
In the case $a_i > n/2$, it is equivalent to the condition of equicomlementability

$$P(a_1, a_2, \ldots, a_k) + \triangle(a_i - n/2) = \triangle(a_1) + \cdots + \triangle(a_{i-1}) + \triangle(a_{i+1}) + \cdots + \triangle(a_k).$$

Thus $P(a_1, a_2, \ldots, a_k)$ and $P(b_1, b_2, \ldots, b_k)$ are scissors congruent when the set $\{a_1, a_2, \ldots, a_k\} = \{b_1, b_2, \ldots, b_k\}$. Now we shall write

$$P(a_1, a_2, \ldots, a_k) \approx P(b_1, b_2, \ldots, b_k),$$

if the set $\{a_1, a_2, \ldots, a_k\} = \{b_1, b_2, \ldots, b_k\}$ and call $P(a_1, a_2, \ldots, a_k)$ is *cake cutting congruent* to $P(b_1, b_2, \ldots, b_k)$. We defined the name *cake cutting congruence*, because of the shape of the following triangles;



Hence cake cutting congruence ($\approx$) is a special kind of scissors congruence by the special way of dissection of the $k-$polygon $P(a_1, a_2, \ldots, a_k) = \triangle(a_1) + \triangle(a_2) + \cdots + \triangle(a_k)$. Thus, for any $k-$polygons F_1 and $F_2 \in \wp_k(n)$, we have

$$F_1 \cong F_2 \Longrightarrow F_1 \approx F_2, \text{ and } F_1 \approx F_2 \Longrightarrow F_1 \sim F_2.$$

The number of equivalent class $\{\wp_k(n)/\approx\}$ is nothing but the number of partitions of $n = a_1 + a_2 + \cdots + a_k$ with $a_i \geq 1$. Thus we know $\#\{\wp_k(n)/\approx\}$ $= p(n-k, k)$.
Let $\zeta = \exp(2\pi i/n)$. Then the area $A(\triangle(a))$ of the triangle $\triangle(a)$ is expressed as $\sin(2a\pi/n)/2 = (\zeta^a - \zeta^{-a})/4i$. Thus we know

$$\begin{array}{rl} & P(a_1, a_2, \ldots, a_k) \sim P(b_1, b_2, \ldots, b_k) \\ \Longleftrightarrow & \sin(2a_1\pi/n) + \cdots + \sin(2a_k\pi/n) = \sin(2b_1\pi/n) + \cdots + \sin(2b_k\pi/n) \\ \Longleftrightarrow & \zeta^{a_1} - \zeta^{-a_1} + \cdots + \zeta^{a_k} - \zeta^{-a_k} = \zeta^{b_1} - \zeta^{-b_1} + \cdots + \zeta^{b_k} - \zeta^{-b_k}. \end{array}$$

In the case $k \geq 4$, we can show the following proposition.

Proposition 2. *For any $k-$polygons F_1 and $F_2 \in \wp_k(n)$ $(k \geq 4)$, we have*

(i) $F_1 \cong F_2 \Longrightarrow F_1 \approx F_2$. *When $n \geq k+2$, there exist $k-$polygons $G_1, G_2 \in \wp_k(n)$ such that $G_1 \approx G_2$ and $G_1 \ncong G_2$.*

(ii) $F_1 \approx F_2 \Longrightarrow F_1 \sim F_2$. *When $n = 4m$ $(m \geq 3)$ and $m \geq k-3$, there exist $k-$polygons $G_1, G_2 \in \wp_k(n)$ such that $G_1 \sim G_2$ and $G_1 \not\approx G_2$.*

Proof. To show (i), we shall take $P(n-k, 2, 1, \ldots, 1), P(n-k, 1, 2, 1, \ldots, 1)$ $\in \wp_k(n)$. Then it is easy to see

$$P(n-k, 2, 1, \ldots, 1) \ncong P(n-k, 1, 2, 1, \ldots, 1), \text{ and}$$

$$P(n-k, 2, 1, \ldots, 1) \approx P(n-k, 1, 2, 1, \ldots, 1).$$

To show (ii), we note that, when n is even, we know $\zeta^{n/2-a} = -\zeta^{-a}$, where $\zeta = \exp(2\pi i/n)$. Hence $\zeta^a - \zeta^{-a} = \zeta^{(n/2-a)} - \zeta^{-(n/2-a)}$. Thus two triangles $\triangle(a)$ and $\triangle(n/2-a)$ have the same area. Hence $\triangle(a) \sim \triangle(n/2-a)$.
When $n = 4m$ $(m \geq 3)$ and $m+3 \geq k$, one can take

$$P(m+1, m+1, m-2, C_1, C_2, \ldots, C_{k-3}) \in \wp_k(n), \text{ and}$$

$$P(m-1, m-1, m+2, C_1, C_2, \ldots, C_{k-3}) \in \wp_k(n).$$

Here $m = C_1 + C_2 + \cdots + C_{k-3}$ is an arbitrary partition of m such as $C_i \geq 1$ for any $1 \leq i \leq k-3$. Then we see

$$\begin{array}{rl} & P(m+1, m+1, m-2, C_1, C_2, \ldots, C_{k-3}) \\ = & \triangle(m+1) + \triangle(m+1) + \triangle(m-2) + \triangle(C_1) + \cdots + \triangle(C_{k-3}) \\ \sim & \triangle(m-1) + \triangle(m-1) + \triangle(m+2) + \triangle(C_1) + \cdots + \triangle(C_{k-3}) \\ = & P(m-1, m-1, m+2, C_1, C_2, \ldots, C_{k-3}). \end{array}$$

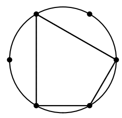
$\{m+1, m+1, m-2\} \neq \{m-1, m-1, m+2\}$ implies that

$P(m{+}1, m{+}1, m{-}2, C_1, C_2, \ldots, C_{k-3}) \not\approx P(m{-}1, m{-}1, m{+}2, C_1, C_2, \ldots, C_{k-3})$,

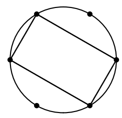
which completes the proof.

Example 1. In the case $n = 6$ and $k = 4$, we have the following example $P(2,2,1,1) \approx P(2,1,2,1)$ and $P(2,2,1,1) \not\cong P(2,1,2,1)$.

$P(2,2,1,1)$



$P(2,1,2,1)$



Example 2. In the case $k = 4$ and $n = 4m + 2$ $(m \geq 2)$, we can find an example such that $P(a_1, a_2, a_3, a_4) \sim P(b_1, b_2, b_3, b_4)$ and $P(a_1, a_2, a_3, a_4) \not\approx P(b_1, b_2, b_3, b_4)$ as follows.

$$\begin{array}{cl} & P(m+1, m+1, m+1, m-1) \\ = & \triangle(m+1) + \triangle(m+1) + \triangle(m+1) + \triangle(m-1) \\ \sim & \triangle(m) + \triangle(m) + \triangle(m) + \triangle(m+2) \\ = & P(m, m, m, m+2), \end{array}$$

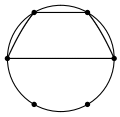
Hence we have $P(m+1, m+1, m+1, m-1) \sim P(m, m, m, m+2)$. From the fact $\{m+1, m+1, m+1, m-1\} \neq \{m, m, m, m+2\}$, we have $P(m+1, m+1, m+1, m-2) \not\approx P(m, m, m, m+2)$.

Example 3. We can find an example $P(a_1, \ldots, a_i) \sim P(b_1, \ldots, b_j)$ with $(3 \leq i < j \leq n$. Actually, take $P(a, a, 3m{-}2a, 3m) \in \wp_4(6m)$ and $P(3m{-}a, 3m{-}a.2a) \in \wp_3(6m)$, where $1 \leq a < 3m/2$. Then we see

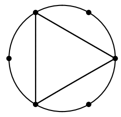
$$\begin{array}{cl} & P(a, a, 3m-2a, 3m) \\ = & \triangle(a) + \triangle(a) + \triangle(3m-2a) \\ \sim & \triangle(3m-a) + \triangle(3m-a) + \triangle(2a) \\ = & P(3m-a, 3m-a, 2a). \end{array}$$

When $m = a = 1$, we have the following figures.

$P(1,1,1,3)$



$P(2,2,2)$



On the other hand, one can also show the following proposition.

Proposition 3. *Assume* $((4k)!, n) = 1$*, then for any* F_1 *and* F_2 *in* $\wp_k(n)$

$$F_1 \approx F_2 \Longleftrightarrow F_1 \sim F_2.$$

Proof. Let $x_1, x_2, \ldots, x_m$ be distinct numbers. X denotes $m \times m$ matrix defined by putting $X = [x_{ij}]$ with $x_{ij} = x_j^i$. Then as an application of Vandermonde determinant, the determinant $|X|$ is equal to

$$x_1 \cdots x_m \prod_{1 \le i < j \le m} (x_j - x_i) \neq 0.$$

Let ζ be $\exp(2\pi i/n)$. Assume

$$P(a_1, a_2, \ldots, a_k) \sim P(b_1, b_2, \ldots, b_k) \text{ but } P(a_1, a_2, \ldots, a_k) \not\approx P(b_1, b_2, \ldots, b_k)$$

Then, from the assumption $P(a_1, a_2, \ldots, a_k) \sim P(b_1, b_2, \ldots, b_k)$, we have

$$\zeta^{a_1} - \zeta^{-a_1} + \cdots + \zeta^{a_k} - \zeta^{-a_k} = \zeta^{b_1} - \zeta^{-b_1} + \cdots + \zeta^{b_k} - \zeta^{-b_k}.$$

Denote $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_k\}$ by $\{c_1, \ldots, c_r\}$. Rearranging the indices if necessary, one can write

$$\{a_1, a_2, \ldots, a_k\} = \{a_1, \ldots, a_1\} \cup \{a_2, \ldots, a_2\} \cup \ldots \cup \{a_s, \ldots, a_s\} \cup \{c_1, \ldots, c_r\},$$

where $n = a_1 + a_2 + \cdots + a_k = e_1 a_1 + e_2 a_2 + \cdots + e_s a_s + c_1 + \cdots c_r$ with $1 \le a_1 < a_2 < \cdots < a_s$. Similarly one can write

$$\{b_1, b_2, \ldots, b_k\} = \{b_1, \ldots, b_1\} \cup \{b_2, \ldots, b_2\} \cup \ldots \cup \{b_t, \ldots, b_t\} \cup \{c_1, \ldots, c_r\},$$

where $n = b_1 + b_2 + \cdots + b_k = f_1 b_1 + f_2 b_2 + \cdots + f_t b_t + c_1 + \cdots c_r$ with $1 \le b_1 < b_2 < \cdots < b_t$. We note that there might exist at most two pairs (i,j) such as $a_i + b_j = n$ $(1 \le i \le s, 1 \le j \le t)$. For such a pair (i,j), we know $(\zeta^{a_i} - \zeta^{-a_i}) - (\zeta^{b_j} - \zeta^{-b_j}) = 2(\zeta^{a_i} - \zeta^{-a_i})$. Thus modifying the indices if necessary, we can write

$$\begin{aligned}&(\zeta^{a_1} - \zeta^{-a_1} + \cdots + \zeta^{a_k} - \zeta^{-a_k}) - (\zeta^{b_1} - \zeta^{-b_1} + \cdots + \zeta^{b_k} - \zeta^{-b_k}) \\ &= e_1(\zeta^{a_1} - \zeta^{-a_1}) + \cdots + e_s(\zeta^{a_s} - \zeta^{-a_s}) - f_1(\zeta^{b_1} - \zeta^{-b_1}) - \cdots - f_t(\zeta^{b_t} - \zeta^{-b_t}).\end{aligned}$$

Here $a_1, n-a_1, \dots, a_s, n-a_s, b_1, n-b_1, \dots, b_t, n-b_t$ are all distinct natural numbers. Moreover $e_1, \dots, e_s, b_1, \dots, f_t$ (≥ 1) satisfy

$$(e_1 + \cdots + e_s) + (f_1 + \cdots + f_t) = 2n,$$

where $e_1+\cdots+e_s = n, n+1$ or $n+2$ according to the existence of the pairs (i,j) such as $a_i + b_j = n$. From the assumption $P(a_1, a_2, \dots, a_k) \not\approx P(b_1, b_2, \dots, b_k)$, $m = 2s + 2t$ satisfies $0 < m \leq 4k$. Put

$$x_1 = \zeta^{a_1}, x_2 = \zeta^{-a_1}, \dots, x_{2s-1} = \zeta^{a_s}, x_{2s} = \zeta^{-a_s}, x_{2s+1} = \zeta^{b_1}, \dots, x_m = \zeta^{-b_t}.$$

b denotes the non-zero vector

$$(e_1, -e_1, \dots, e_s, -e_e, -f_1, f_1, \dots, -f_t, f_t) \in \mathbb{Z}^m.$$

We note that, from the assumption $((4k)!, n) = 1$, $\sigma_i : \zeta \to \zeta^i$ implies an automorphism σ_i of $Gal(\mathbb{Q}(\zeta)/\mathbb{Q})$ for any $2 \leq i \leq 4k$. Since $(x_1, x_2, \dots, x_m)^t b = 0$ and $\sigma_i(x_j) = x_j^i$, we know $(x_1^i, x_2^i \dots, x_m^i)^t b = 0$ for any $1 \leq i \leq m$. Hence we must have

$$X^t b = \mathbf{0},$$

for the regular matrix $X = [x_j^i]$, which contradicts $b \neq \mathbf{0}$. Thus we have completed the proof.

Combining Proposition 2 and Proposition 3, we have shown the following theorem.

Theorem 1. *Let $k \geq 4$. Then for any $k-$polygons F_1 and F_2 in $\wp_k(n)$ we have the following.*
(i) *$F_1 \cong F_2 \Longrightarrow F_1 \approx F_2$. When $n \geq k+2$, there exist $k-$polygons $G_1, G_2 \in \wp_k(n)$ which are cake cutting congruent but not congruent.*
(ii) *$F_1 \approx F_2 \Longrightarrow F_1 \sim F_2$. When $n = 4m$, there exist $k-$polygons $G_1, G_2 \in \wp_k(n)$ which are scissors congruent but not cake cutting congruent.*
(iii) *In the case $(4k)!, n) = 1$, two polygons in $\wp_k(n)$ are scissors congruent if and only if they are cake cutting congruent.*

In the following, we shall study the special case $\wp_3(n)$ more precisely. In our previous papers [3] and [4], we have proved $C_3(b) = \#\{\wp_3(n)/\cong\} = p(n-3,3) = \#\{\wp_3(n)/\approx\}$. Hence we have already proved the following proposition.

Proposition 4. *For any triangles in $\wp_3(n)$,*

$$\begin{aligned} & P(a_1, b_1, c_1) \cong P(a_2, b_2, c_2) \Longleftrightarrow P(a_1, b_1, c_1) \approx P(a_2, b_2, c_2) \\ \Longleftrightarrow \quad & \{a_1, b_1, c_1\} = \{a_2, b_2, c_2\}. \end{aligned}$$

As a corollary of Proposition 3, we have:

Corollary 1. *When $n(\geq 3)$, where n is coprime to $2310(= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)$, for any triangles in $\wp_3(n)$, we have*

$$P(a_1, b_1, c_1) \approx P(a_2, b_2, c_2) \Longleftrightarrow P(a_1, b_1, c_1) \sim P(a_2, b_2, c_2).$$

On the contrary, we can show the following proposition.

Proposition 5. *When $n = 4m$ with $m \geq 3$, there exists two triangles $P(a_1, b_1, c_1), P(a_2, b_2, c_2) \in \wp_3(n)$ such as $P(a_1, b_1, c_1) \sim P(a_2, b_2, c_2)$ but $P(a_1, b_1, c_1) \not\approx P(a_2, b_2, c_2)$.*

Proof. Take a natural number a which satisfies $0 < a < m$ and $2a \neq m$. Then $\{2a, m - a, 3m - a\} \neq \{a, 2m + a, 2m - 2a\}$, i.e.,
$P(2a, m - a, 3m - a) \not\approx P(a, 2m + a, 2m - 2a)$.
We know

$$P(2a, m - a, 3m - a) = \triangle(2a) + \triangle(m - a) + \triangle(3m - a),$$

and

$$P(a, 2m + a, 2m - 2a) = \triangle(a) + \triangle(2m + a) + \triangle(2m - 2a),$$

where $2m < 3m - a < 4m$ and $2m < 2m + a < 4m$. Thus

$$P(2a, m - a, 3m - a) + \triangle(m - a) + \triangle(a) = \triangle(2a) + \triangle(m - a) + \triangle(a),$$

and

$$P(a, 2m + a, 2m - 2a) + \triangle(m - a) + \triangle(a) = \triangle(a) + \triangle(m - a) + \triangle(2m - 2a)$$

$\sim \triangle(2a) + \triangle(m - a) + \triangle(a)$.
Therefore $P(2a, m-a, 3m-a)$ is equicomplementable to $P(a, 2m+a, 2m-2a)$, which completes the proof.

Combining Proposition 4 and 5, we have shown the following theorem.

Theorem 2. *For any triangles $F_1, F_2 \in \wp_3(n)$,*

$$F_1 \cong F_2 \Longleftrightarrow F_1 \approx F_2, \text{ and } F_1 \approx F_2 \Longrightarrow F_1 \sim F_2, \text{ in general.}$$

(i) *In the case $(n, 2310) = 1$, $F_1 \approx F_2 \Longleftrightarrow F_1 \sim F_2$.*

(ii) *In the case $n = 4m$ with $m \geq 3$, there exist F_1, F_2 which are scissors congruent but $F_1 \not\approx F_2$.*

As a corollary of this theorem, we can show:

Corollary 2. *Let p be an odd prime. Then for any triangles in $\wp_3(p)$,*

$$P(a_1,b_1,c_1)\approx P(a_2,b_2,c_2)\Longleftrightarrow P(a_1,b_1,c_1)\sim P(a_2,b_2,c_2).$$

Proof. We note that the assertion has been already proved for the cases $p\geq 13$ in Corollary 1. In the remaining cases $p=3,5,7,11$, $P(a_1,b_1,c_1)\not\sim P(a_2,b_2,c_2)$ for any triangles $P(a_1,b_1,c_1)\not\cong P(a_2,b_2,c_2)$ can be proved as follows.

Since $\#\{\wp_3(p)/\cong\}=C_3(p)=\left\{\dfrac{p^2}{12}\right\}$ as was shown in [3], we have $C_3(5)=2, C_3(7)=4, C_3(11)=10$. More precisely we have

$$\begin{aligned}
5 &= 1+1+3=1+2+2,\\
7 &= 1+1+5=1+2+4=1+3+3=2+2+3,\\
11 &= 1+1+9=1+2+8=1+3+7=1+4+6=1+5+5\\
&= 2+2+7=2+3+6=2+4+5=3+3+5=3+4+4.
\end{aligned}$$

From Proposition 1, we know $P(a_1,b_1,c_1)\sim P(a_2,b_2,c_2)$ if and only if $\sin(2a_1\pi/p)+\sin(2b_1\pi/p)+\sin(2c_1\pi/p)=\sin(2a_2\pi/p)+\sin(2b_2\pi/p)+\sin(2c_2\pi/p)$. One can easily verify if $P(a_1,b_1,c_1)\not\cong P(a_2,b_2,c_2)$, then $A(P(a_1,b_1,c_1))\neq A(P(a_2,b_2,c_2))$ as follows:

p	(a,b,c)	$2A(P(a,b,c))=\sin(2a\pi/p)+\sin(2b\pi/p)+\sin(2c\pi/p)$
5	$(1,1,3)$	$1.314327780297834\ldots$
	$(1,2,2)$	$2.126627020880009\ldots$
7	$(1,1,5)$	$0.588735052754236\ldots$
	$(1,2,4)$	$1.322875655532295\ldots$
	$(1,3,3)$	$1.649598960703146\ldots$
	$(2,2,3)$	$2.383739563481205\ldots$
11	$(1,1,9)$	$0.171649639556676\ldots$
	$(1,2,8)$	$0.460451370929183\ldots$
	$(1,3,7)$	$0.774712684982272\ldots$
	$(1,4,6)$	$1.014657834968426\ldots$
	$(1,5,5)$	$1.104105931138457\ldots$
	$(2,2,7)$	$1.063514416354778\ldots$
	$(2,3,6)$	$1.617720880394021\ldots$
	$(2,4,5)$	$1.947114126550206\ldots$
	$(3,3,5)$	$2.261375440603295\ldots$
	$(3,4,4)$	$2.501320590589449\ldots$

Generating functions

In this section, we shall investigate the generating functions related to $\wp_k(n)$. Since $\#\wp_k(n) = \left(\begin{array}{c} n \\ k \end{array} \right)$, from the binomial series theorem, we have

$$\sum_{n=0}^{\infty} (\#\wp_k(n)) q^n = \sum_{n=0}^{\infty} \left(\begin{array}{c} n \\ k \end{array} \right) q^n = \frac{q^k}{(1-q)^{k+1}}.$$

In our previous papers [3] and [4], we have investigated the number of incongruent $k-$polygons in $\wp_k(n)$, i.e., $C_k(n) = \#\{\wp_k(n)/\cong\}$ for small k. Here we shall recall several results obtained in [3] and [4] as follows.

Let $p(n, m)$ be the number of partitions of n with each part $\leq m$, that is,

$$p(n, m) = p(n \mid \text{parts in } \{1, 2, \ldots, m\}).$$

Then $p(n, 1) = 1$ and $p(n, 2) = \left[\frac{n}{2} \right] + 1$, where $[x]$ denotes the greatest integer $\leq x$. Moreover $p(n, 3) = \left\{ \frac{(n+3)^2}{12} \right\}$, where $\{x\}$ denotes the nearest integer to x. The generating function of $p(n, m)$ is:

$$\sum_{n=0}^{\infty} p(n, m) q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

We have shown the following results on $C_k(n)$ for $k = 3, 4$ and 5 in [3] and [4].

Proposition 6 ([3]). *$C_3(n) = p(n-3, 3) = \left\{ \frac{n^2}{12} \right\}$, and the generating function of $C_3(n)$ is*

$$\sum_{n=0}^{\infty} C_3(n) q^n = \frac{q^3}{(1-q)(1-q^2)(1-q^3)}.$$

Proposition 7 ([4]). *$C_4(n) = p(n-4, 4) + p(n-6, 4) + p(n-8, 4)$, and the generating function of $C_4(n)$ is*

$$\sum_{n=0}^{\infty} C_4(n) q^n = \frac{q^4 + q^6 + q^8}{(1-q)(1-q^2)(1-q^3)(1-q^4)}.$$

Proposition 8 ([4]). *The generating function of $C_5(n)$ is given by*

$$\sum_{n=0}^{\infty} C_5(n)q^n = \frac{q^5+q^7+q^8+2q^9+2q^{10}+2q^{11}+q^{12}+q^{13}+q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

It seems difficult to obtain the generating functions of $C_k(n)$ for general k, but for $\#\{\wp_k(n)/\approx\} = p(n-k,k)$, we have in general

$$\sum_{n=0}^{\infty}(\#\{\wp_k(n)/\approx\})q^n = \sum_{n=0}^{\infty} p(n-k,k)q^n = \frac{q^k}{(1-q)\cdots(1-q^k)}.$$

It seems also difficult to obtain the generating function of $\#\{\wp_k(n)/\sim\}$.

Here we shall give a table of $\#\{\wp_3(n)/\sim\}$ for small values $n \leq 20$. In the following table, the symbol $(*)$ expresses the value n for which $\#\{\wp_3(n)/\sim\} \neq \#\{\wp_3(n)/\approx\}$.

n	$\#\{\wp_3(n)/\sim\}$	$C_3(n) = \left\{\frac{n^2}{12}\right\}$	n	$\#\{\wp_3(n)/\sim\}$	$C_3(n) = \left\{\frac{n^2}{12}\right\}$
3	1	1	12	11 $(*)$	12
4	1	1	13	14	14
5	2	2	14	16	16
6	3	3	15	19	19
7	4	4	16	20 $(*)$	21
8	5	5	17	24	24
9	7	7	18	27	27
10	8	8	19	30	30
11	10	10	20	31 $(*)$	33

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